

DID THE EVER DEAD OUTNUMBER THE LIVING AND WHEN? A BIRTH-AND-DEATH APPROACH

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ABSTRACT. This paper is an attempt to formalize analytically the question raised in “World Population Explained: Do Dead People Outnumber Living, Or Vice Versa?” Huffington Post, [7]. We start developing simple deterministic Malthusian growth models of the problem (with birth and death rates either constant or time-dependent) before running into both linear birth and death Markov chain models and age-structured models.

Keywords: population growth; Malthusian; constant vs time-dependent birth/death rates; time-inhomogeneous Markov chain; age-structured models.

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1. INTRODUCTION

Starting from a question raised in (of all places) Huffington Post, asking whether the total number of dead people ever outgrew the total number of living people at a certain time, and when the crossover (if any) occurred, leads us to analyze some simplified models of population growth based on “mean-field” approximations, i.e. dealing with the notion of mean birth/death rates.

We shall successively consider a simple deterministic growth model for the whole population (Section 2); a linear birth/death Markov chain model for the growth of the whole population (Section 3); a deterministic age-structured model (Lotka-McKendrick-von Foerster) in Section 4, for which we shall explicitly (as far as possible) derive general solutions for the population sizes (in all 3 cases), the extinction probability, time to extinction and joint law for the number of people ever born and ever dead (for the Markovian process). Some explicitly solvable cases will be considered.

2. SIMPLE BIRTH AND DEATH MODELS FOR POPULATION GROWTH (DETERMINISTIC)

We get first interested in the simple deterministic evolution of a population whose individuals are potentially immortal, non-interacting and living in a bath with unlimited resources.

2.1. Constant rates (Malthusian model). Let the mean size of some population at time $t = 0$ be $x(0) = x_0$ and let $x(t)$ be its mean size at time t .

Let (λ_b, λ_d) be the birth and death rates per individual, assumed constant in the first place. With $\dot{\cdot} = d/dt$, consider the evolution

$$(1) \quad \dot{x}(t) = (\lambda_b - \lambda_d)x(t) =: \dot{x}_b(t) - \dot{x}_d(t),$$

where $\dot{x}(t)$ is the growth rate at time t of the population, whereas $\dot{x}_b(t) = \lambda_b x(t)$ and $\dot{x}_d(t) = \lambda_d x(t)$ are the birth and death rates at time t of this population (the rates at which newborns and dead people are created, respectively).

Integrating, we have $x(t) = x_b(t) - x_d(t)$, with $x(0) = x_b(0)$. The quantities $x_b(t)$ and $x_d(t)$ are the mean number of individuals ever born and ever dead between times 0 and t and therefore $x(t)$ is the current population size ($x_b(t)$ includes the initial individuals). Obviously, with $\lambda := \lambda_b - \lambda_d$

$$x(t) = x(0) e^{\lambda t}$$

and therefore (if $\lambda_b \neq \lambda_d$)

$$\begin{aligned} x_b(t) &= x(0) + \frac{\lambda_b}{\lambda_b - \lambda_d} (x(t) - x(0)) \\ x_d(t) &= \frac{\lambda_d}{\lambda_b - \lambda_d} (x(t) - x(0)). \end{aligned}$$

Under this model, given that some individual is alive at time t , its lifetime τ_d obeys $\mathbf{P}(\tau_d > t + \tau \mid \tau_d > t) = \mathbf{P}(\tau_d > \tau) = e^{-\lambda_d \tau}$, independent of t (the memory-less property of the exponential distribution). The individuals are immortal in that there is no upper bound for their lifetime although very long lifetimes are unlikely to occur, i.e. with exponentially small probabilities.

Similarly, its reproduction time τ_b obeys $\mathbf{P}(\tau_b > t + \tau \mid \tau_b > t) = \mathbf{P}(\tau_b > \tau) = e^{-\lambda_b \tau}$. After each reproduction time, each individual is replaced by two individuals by binary splitting. Then λ_d^{-1} and λ_b^{-1} are the mean lifetime and reproduction times.

We assume the supercritical condition: $\lambda = \lambda_b - \lambda_d > 0$ so that $x(t)$ grows exponentially fast. Then, as $t \rightarrow \infty$

$$(2) \quad \frac{x_b(t)}{x(t)} \rightarrow \frac{\lambda_b}{\lambda_b - \lambda_d} \text{ and } \frac{x_d(t)}{x(t)} \rightarrow \frac{\lambda_d}{\lambda_b - \lambda_d}.$$

Remark: The cases $\lambda_b = \lambda_d$ or $\lambda_b - \lambda_d < 0$ correspond to the critical and subcritical situations, respectively. In the first case, the population size $x(t)$ remains constant (while $x_b(t)$ and $x_d(t)$ grow linearly) whereas in the second case, the population size goes to 0 exponentially fast. Unless otherwise specified we shall stick to the supercritical case in the sequel.

• Suppose we ask the questions:

- how many people have ever lived in the past and
- do the ever dead outnumber the living and when?

The answer to the first question is $x_b(t)$.

The answer to the second question is positive iff $\lambda_b > \lambda_d$ and $\frac{\lambda_d}{\lambda_b - \lambda_d} > 1$ or else if $\lambda_d < \lambda_b < 2\lambda_d$. Then the time when the overshooting occurred is t_* defined by

$\frac{x_d(t_*)}{x(t_*)} = 1$. Thus

$$t_* = -\frac{1}{\lambda} \log \left(2 - \frac{\lambda_b}{\lambda_d} \right) > 0,$$

which is independent of $x(0)$. At time t_* , the number of people who have ever lived in the past is thus

$$x_b(t_*) = x(0) \left(1 + \frac{\lambda_b}{\lambda_b - \lambda_d} (e^{\lambda t_*} - 1) \right) = x(0) \frac{2\lambda_d}{2\lambda_d - \lambda_b} > x(0).$$

Note also that the population size at t_* is

$$x(t_*) = x(0) e^{\lambda t_*} = x(0) \frac{\lambda_d}{2\lambda_d - \lambda_b} > x(0),$$

so twice less than $x_b(t_*)$ consistently with $x(t_*) = x_b(t_*) - x_d(t_*)$ and $x_d(t_*) = x(t_*)$.

Suppose that at some (large) terminal time t_f we know:

- $x_b(t_f)$ the number of people who ever lived before time t_f .
- the initial population size $x(0)$ at time $t = 0$.
- the current population size $x(t_f)$.

Then

$$\lambda = \lambda_b - \lambda_d = \frac{1}{t_f} \log \left(\frac{x(t_f)}{x(0)} \right) \text{ and } \frac{\lambda_b}{\lambda_b - \lambda_d} \sim \frac{x_b(t_f)}{x(t_f)}$$

and both λ_b and λ_d are known. As a result,

$$t_* = -\frac{1}{\lambda} \log \left(2 - \frac{\lambda_b}{\lambda_d} \right) \sim t_f \frac{\log(1 + x(t_f)/(x_b(t_f) - 2x(t_f)))}{\log(x(t_f)/x(0))}$$

can be estimated from the known data.

Example:

The number of people who ever lived on earth is estimated to $x_b(t_f) = 105.10^9$. Suppose 10.000 years ago the initial population was of 10^6 units. Then $x(0) = 10^6$ and $t_f = 10^4$. The current world population is around $x(t_f) = 7.10^9$. Then

$$\lambda_b \sim 15.10^{-3}, \lambda_d \sim 14.10^{-3} \text{ and } t_* \sim 75 \text{ years.}$$

These rough estimates based on most simple Malthus growth models are very sensitive to the initial condition. \square

2.2. Time-dependent rates. Because the assumption of constant birth and death rates is not a realistic hypothesis, we now allow time-dependent rates, so both λ_b and λ_d are now assumed to depend on current time t . We shall assume that $\lambda_b(t)$ is bounded above by $\lambda_b^+ < \infty$ and below by $\lambda_b^- > 0$ and similarly for $\lambda_d(t)$ with upper and lower bounds λ_d^\pm . Unless specified otherwise, we assume supercriticality throughout, that is $\lambda_b^- > \lambda_d^-$. We shall also assume that both $\lambda_b(t)$ and $\lambda_d(t)$ are nonincreasing functions of t .¹

Let $\Lambda_b(t) = \int_0^t ds \lambda_b(s)$, $\Lambda_d(t) = \int_0^t ds \lambda_d(s)$ be primitives of $\lambda_b(t)$ and $\lambda_d(t)$.

¹Under this hypothesis, both the expected lifetime and the mean reproduction time of each individual increase as time passes by, reflecting an improvement of the general survival conditions.

Under this model, given some individual is alive at time t , its lifetime τ_d obeys $\mathbf{P}(\tau_d > t + \tau \mid \tau_d > t) = e^{-(\Lambda_d(t+\tau) - \Lambda_d(t))}$ so $\mathbf{P}(\tau_d \in dt \mid \tau_d > t) = \lambda_d(t) dt$.

Similarly, its reproduction time τ_b obeys $\mathbf{P}(\tau_b > t + \tau \mid \tau_b > t) = e^{-(\Lambda_b(t+\tau) - \Lambda_b(t))}$ so $\mathbf{P}(\tau_b \in dt \mid \tau_b > t) = \lambda_b(t) dt$. After each reproduction time, each individual is again replaced by two individuals, giving birth to a new individual by binary splitting.

With $\dot{\bullet} \equiv d/dt$, we thus have to consider the new evolution

$$(3) \quad \dot{x}(t) = (\lambda_b(t) - \lambda_d(t)) x(t) =: \dot{x}_b(t) - \dot{x}_d(t),$$

where $\dot{x}(t)$ is the growth rate at time t of the population, whereas $\dot{x}_b(t) = \lambda_b(t) x(t)$ and $\dot{x}_d(t) = \lambda_d(t) x(t)$ are again the birth and death rates at time t of this population.

We immediately have $x(t) = x_b(t) - x_d(t)$, with $x(0) = x_b(0)$. The quantities $x_b(t)$ and $x_d(t)$ are the mean number of individuals ever born and ever dead between times 0 and t ($x_b(t)$ including the initial individuals) and therefore $x(t)$ is the current population size. Clearly, with $\Lambda(t) = \Lambda_b(t) - \Lambda_d(t)$

$$x(t) = x(0) e^{\Lambda(t)}$$

and therefore

$$\begin{aligned} x_b(t) &= x(0) \left(1 + \int_0^t ds \lambda_b(s) e^{\Lambda(s)} \right) \\ x_d(t) &= x(0) \int_0^t ds \lambda_d(s) e^{\Lambda(s)}. \end{aligned}$$

$$\text{Note } x(t) = x(0) \left(1 + \int_0^t ds (\lambda_b(s) - \lambda_d(s)) e^{\Lambda(s)} \right)$$

We assume that $\Lambda(t)$ is non-decreasing with t so that $x(t)$ grows and also that, with $\lambda_b^- > \lambda_d^- > 0$

$$\lambda_b(t) \rightarrow \lambda_b^- > 0 \text{ and } \lambda_d(t) \rightarrow \lambda_d^- > 0 \text{ as } t \rightarrow \infty.$$

Then, as $t \rightarrow \infty$

$$(4) \quad \begin{aligned} \frac{x_b(t)}{x(t)} &= \frac{1 + \int_0^t ds \lambda_b(s) e^{\Lambda(s)}}{1 + \int_0^t ds (\lambda_b(s) - \lambda_d(s)) e^{\Lambda(s)}} \rightarrow \frac{\lambda_b^-}{\lambda_b^- - \lambda_d^-} \text{ and} \\ \frac{x_d(t)}{x(t)} &= \frac{\int_0^t ds \lambda_d(s) e^{\Lambda(s)}}{1 + \int_0^t ds (\lambda_b(s) - \lambda_d(s)) e^{\Lambda(s)}} \rightarrow \frac{\lambda_d^-}{\lambda_b^- - \lambda_d^-}. \end{aligned}$$

Indeed, by L'Hospital rule, if $\frac{f(t)}{g(t)} \xrightarrow{t \rightarrow \infty} c \Rightarrow \frac{F(t)}{G(t)} \xrightarrow{t \rightarrow \infty} c$ where (F, G) are the primitives of (f, g) . Applying L'Hospital rule to $f = \lambda_b(t) e^{\Lambda(t)}$ and $g = (\lambda_b(t) - \lambda_d(t)) e^{\Lambda(t)}$ gives the result. Suppose $\lambda_b^- > \lambda_d^-$. Whenever $\frac{\lambda_d^-}{\lambda_b^- - \lambda_d^-} > 1$ or else $\lambda_b^- < 2\lambda_d^-$, there is a unique t_* such that $\frac{x_d(t_*)}{x(t_*)} = 1$.

Remark: Assume $\lambda_b^- > \lambda_d^- > 0$. We have the detailed balance equations:

$$\frac{d}{dt} \begin{bmatrix} x_b(t) \\ x_d(t) \end{bmatrix} = \begin{bmatrix} \lambda_b(t) & -\lambda_b(t) \\ \lambda_d(t) & -\lambda_d(t) \end{bmatrix} \begin{bmatrix} x_b(t) \\ x_d(t) \end{bmatrix}, \text{ with } \begin{bmatrix} x_b(0) \\ x_d(0) \end{bmatrix} = \begin{bmatrix} x_b(0) \\ 0 \end{bmatrix}.$$

In vector form, this is also $\dot{\mathbf{x}}(t) = B(t) \mathbf{x}(t)$, $\mathbf{x}(0)$ where $\mathbf{x}(t) \succeq \mathbf{0}$ (componentwise). The matrix $B(t)$ has eigenvalues $(0, \lambda_b(t) - \lambda_d(t))$, with $(0, \lambda_b(t) - \lambda_d(t)) \rightarrow (0, \lambda_b^- - \lambda_d^-)$ as $t \rightarrow \infty$. $\lambda_1(t) = \lambda_b(t) - \lambda_d(t)$ is its dominant eigenvalue. Letting $X(t)$ be a 2×2 matrix with $X(0) = I$, and $\dot{X}(t) = B(t)X(t)$, we have $\mathbf{x}(t) = X(t) \mathbf{x}(0)$ and

$$e^{-\int_0^t \lambda_1(s) ds} X(t) \rightarrow \frac{\mathbf{x} \mathbf{y}'}{\mathbf{x}' \mathbf{y}} \text{ as } t \rightarrow \infty,$$

with $B(\infty) \mathbf{x} = \theta \mathbf{x}$, $\mathbf{y}' B(\infty) = \theta \mathbf{y}'$, $\theta = \lambda_b^- - \lambda_d^- := \lambda_1(\infty)$ the dominant eigenvalue of $B(\infty)$. The left and right eigenvectors of $B(\infty)$ associated to θ are found to be $\mathbf{x}' = (\lambda_b^-, \lambda_d^-)$ and $\mathbf{y}' = (1, -1)$. The latter convergence result holds because, with \mathbf{y}' independent of t , $\mathbf{y}' B(t) = \lambda_1(t) \mathbf{y}'$ for all $t > 0$ and $B(t) \mathbf{1} = \mathbf{0}$ ($\mathbf{1}$ is in the kernel of $B(t)$ for all t), see [2]. We are thus led to

$$\begin{aligned} e^{-\int_0^t \lambda_1(s) ds} X(t) &= e^{-\Lambda(t)} X(t) \rightarrow \frac{1}{\lambda_b^- - \lambda_d^-} \begin{bmatrix} \lambda_b^- & -\lambda_b^- \\ \lambda_d^- & -\lambda_d^- \end{bmatrix} \text{ as } t \rightarrow \infty \\ \text{and } e^{-\Lambda(t)} \mathbf{x}(t) &\rightarrow \frac{x(0)}{\lambda_b^- - \lambda_d^-} \begin{bmatrix} \lambda_b^- \\ \lambda_d^- \end{bmatrix} \text{ as } t \rightarrow \infty, \end{aligned}$$

which is consistent with the previous analysis making use of L'Hospital rule, recalling $x(t) = x_b(t) - x_d(t) = x(0) e^{\Lambda(t)}$. Note

$$\frac{1}{t} \int_0^t \lambda_1(s) ds \rightarrow \theta \text{ as } t \rightarrow \infty.$$

Examples:

(i) (homographic rates) With $b > a > 0$ and $d > c > 0$, assume

$$\lambda_b(t) = \frac{at+b}{t+1} \text{ and } \lambda_d(t) = \frac{ct+d}{t+1}$$

Thus $\lambda_b^- = a$, $\lambda_b^+ = b$ and $\lambda_d^- = c$, $\lambda_d^+ = d$. Then

$$\Lambda_b(t) = at + (b-a) \log(t+1) \text{ and } \Lambda_d(t) = ct + (d-c) \log(t+1)$$

and with $\delta = (b-a) - (d-c)$

$$\Lambda(t) = (a-c)t + \delta \log(t+1) \text{ and } x(t) = x(0) (t+1)^\delta e^{(a-c)t}.$$

Note $\delta \geq 0$, depending on $b \geq d$. Thus

$$\begin{aligned} x_b(t) &= x(0) \left(1 + \int_0^t ds (as+b) (s+1)^{\delta-1} e^{(a-c)s} \right) \\ x_d(t) &= x(0) \int_0^t ds (cs+d) (s+1)^{\delta-1} e^{(a-c)s} \end{aligned}$$

and $\frac{x_d(t)}{x(t)} \xrightarrow{t \rightarrow \infty} \frac{c}{a-c}$. When $a > c$, $\log x(t)/t \rightarrow a-c > 0$ and the population grows at exponential rate.

If in addition $a < 2c$, there is a unique t_* such that $x_d(t_*) = x(t_*)$:

$$\int_0^{t_*} ds (cs+d) (s+1)^{\delta-1} e^{(a-c)s} = (t_*+1)^\delta e^{(a-c)t_*}.$$

(ii) Gompertz (a critical case $\lambda_b^- = a = c = \lambda_d^-$): We briefly illustrate here on an example that population growth under criticality also has a rich structure. Let

$$\lambda_b(t) = a + (b - a)e^{-\alpha t} \text{ and } \lambda_d(t) = c + (d - c)e^{-\alpha t}.$$

Assume $\alpha > 0$, $a = c$ and $b - d > \alpha$. Then

$$\begin{aligned} \Lambda_b(t) &= at + \frac{b-a}{\alpha}(1 - e^{-\alpha t}) \text{ and } \Lambda_b(t) = ct + \frac{d-c}{\alpha}(1 - e^{-\alpha t}) \\ \Lambda(t) &= \Lambda_b(t) - \Lambda_d(t) = \frac{b-d}{\alpha}(1 - e^{-\alpha t}). \end{aligned}$$

Because $\lambda_b^- = a = c = \lambda_d^-$, $x_d(t)$ and $x(t)$ will never meet.

We have $x(t) = x(0)e^{\Lambda(t)} = x(0)e^{(b-d)(1-e^{-\alpha t})/\alpha} \rightarrow x(0)e^{(b-d)/\alpha}$ as $t \rightarrow \infty$: due to $b-d > \alpha$, after an initial early time of exponential growth, the population size stabilizes to a limit. This model bears some resemblance with the time-homogeneous logistic (or Verhulst) growth model with alternation of fast and slow growth and an inflection point between the two regimes. \square

3. BIRTH AND DEATH MARKOV CHAIN (STOCHASTICITY)

The deterministic dynamics discussed for $(x(t), x_b(t), x_d(t))$ arise as the mean values of some continuous-time stochastic integral-valued birth and death Markov chain for $(N(t), N_b(t), N_d(t))$ which we would like now to discuss; $N_b(t), N_d(t)$ are now respectively the number of people who ever lived and died at time t to the origin, while $N(t) = N_b(t) - N_d(t)$ is the number of people currently alive at t . Due to stochasticity, some new effects are expected to pop in, especially the possibility of extinction, even in the supercritical regime.

3.1. The current population size $N(t)$.

3.1.1. *Probability generating function (pgf) of $N(t)$.* Let $N(t) \in \{0, 1, 2, \dots\}$ with $N(0) \stackrel{d}{\sim} \pi_0$ for some given initial probability distribution π_0 . Define the transition matrix of a (linear) birth and death Markov chain [9] as

$$\begin{aligned} (5) \quad \mathbf{P}(N(t+dt) = n+1 \mid N(t) = n) &= \lambda_b(t)ndt + o(dt) \\ \mathbf{P}(N(t+dt) = n-1 \mid N(t) = n) &= \lambda_d(t)ndt + o(dt), \end{aligned}$$

with state $\{0\}$ therefore absorbing. Let $\phi_t(z) = \mathbf{E}\mathbf{E}(z^{N(t)} \mid N(0)) =: \mathbf{E}(z^{N(t)})$ be the pgf of $N(t)$ averaged over $N(0)$. Let $\phi_0(z) = \mathbf{E}(z^{N(0)})$ be the pgf of $N(0)$. Then, defining

$$g(t, z) = (\lambda_b(t) - \lambda_d(t))(z-1) + \lambda_b(t)(z-1)^2,$$

$\phi_t(z)$ obeys the PDE (see [1])

$$\partial_t \phi_t(z) = g(t, z) \partial_z \phi_t(z), \text{ with } \phi_{t=0}(z) = \phi_0(z)$$

or equivalently, solves the non-linear Bernoulli ODE problem

$$\dot{\varphi}_t(z) = -g(t, \varphi_t(z)), \text{ with } \varphi_{t=0}(z) = z \text{ and } \phi_t(z) = \phi_0(\varphi_t^{-1}(z)).$$

Setting $\psi(t) = e^{\Lambda(t)}$ and $\eta(t) = \int_0^t ds \lambda_b(s) e^{\Lambda(t)-\Lambda(s)}$, both going to ∞ as $t \rightarrow \infty$, assuming

$$\lambda_b(t) \rightarrow \lambda_b^- > 0 \text{ and } \lambda_d(t) \rightarrow \lambda_d^- > 0 \text{ as } t \rightarrow \infty$$

and $\lambda_b^- \neq \lambda_d^-$, we easily get the solution

$$(6) \quad \phi_t(z) = \phi_0 \left(\frac{1 - (\eta(t) - \psi(t))(z-1)}{1 - \eta(t)(z-1)} \right),$$

where inside $\phi_0(\cdot)$ we recognize $\varphi_t(z)$ as an homographic pgf, solution to the ODE problem. We choose for initial condition a ‘Bernoulli thinned’ geometric random variable with homographic pgf²

$$\phi_0(z) = \frac{q(q_0 + p_0 z)}{1 - p(q_0 + p_0 z)} = \frac{q + p_0 q(z-1)}{q - p_0 p(z-1)};$$

this considerably simplifies (6), since the composition of two homographic pgfs is again an homographic pgf. With this initial condition, we finally get

$$\phi_t(z) = \frac{q - (q\eta(t) - p_0 q \psi(t))(z-1)}{q - (q\eta(t) + p_0 p \psi(t))(z-1)}.$$

With $x(0) = \phi'_0(1) = p_0/q > 0$ (with ‘ ’ the derivative with respect to z), averaging over $N(0)$, we obtain

$$\begin{aligned} x(t) &= \mathbf{E}(N(t)) = \mathbf{E}\mathbf{E}(N(t) | N(0)) = \phi'_t(1) = \phi'_0(1)\psi(t) = x(0)e^{\Lambda(t)} \\ \text{Var}(N(t)) &= \phi''_t(1) + \phi'_t(1) - \phi'_t(1)^2 = x(0)^2(2p-1)\psi(t)^2 + x(0)\psi(t)(1+2\eta(t)). \end{aligned}$$

Remark: For pgfs of the homographic form (6), it is easy to see that, averaging over $N(0)$, for $n \geq 1$ ³

$$\begin{aligned} \mathbf{P}(N(t) = n) &= \mathbf{E}\mathbf{P}(N(t) = n | N(0)) \\ &= [z^n] \phi_t(z) = C(t)^{n-1} (\phi_t(0)C(t) + D(t)), \end{aligned}$$

where, with $\phi'_t(0) = \mathbf{P}(N(t) = 1) = \phi'_0(0)\psi(t)(1+\eta(t))^{-2}$

$$C(t) := 1 - \frac{\phi'_t(0)}{1 - \phi_t(0)} \text{ and } D(t) := \frac{\phi'_t(0)}{1 - \phi_t(0)} - \phi_t(0).$$

3.1.2. *The extinction probability and time till extinction.* The extinction probability reads

$$\mathbf{P}(N(t) = 0) = \phi_t(0) = 1 - \frac{p_0 \psi(t)}{q + q\eta(t) + p_0 p \psi(t)}.$$

Because $\mathbf{P}(N(t) = 0)$ is also $\mathbf{P}(\tau_e \leq t)$ where τ_e is the extinction time (the event $\tau_e = \infty$ corresponds to non-extinction), we obtain

$$\mathbf{P}(\tau_e > t) = \frac{p_0 \psi(t)}{q + q\eta(t) + p_0 p \psi(t)}.$$

Whenever $\lambda_b^- > \lambda_d^- > 0$ (supercritical regime), $\eta(t)/\psi(t) \rightarrow \kappa > 1$ (as $t \rightarrow \infty$) because

$$\eta(t)/\psi(t) = \int_0^t ds \lambda_b(s) e^{-\Lambda(s)} > \int_0^t ds \lambda(s) e^{-\Lambda(s)} = 1 - e^{-\Lambda(t)} \rightarrow 1.$$

² $\mathbf{E}(z^{N(0)}) := \phi_0(z) = \phi_G(\phi_B(z))$ is the composition of a geometric pgf $\phi_G(z) = (qz)/(1-pz)$ ($p+q=1$) with a Bernoulli pgf $\phi_B(z) = q_0 + p_0 z$ ($p_0+q_0=1$). So $N(0) \stackrel{d}{=} \sum_{i=1}^G B_i$ where the B_i s are iid Bernoulli, independent of G with $\mathbf{E}(N(0)) = p_0/q$ and $\text{Var}(N(0)) = p_0(q_0 q + p_0 p)/q^2 = q_0 \mathbf{E}(N(0)) + p \mathbf{E}(N(0))^2$.

³ $[z^n] \phi_t(z)$ is the coefficient of z^n in the series expansion of $\phi_t(z)$.

As it can be checked, were both $\lambda_b(t)$ and $\lambda_d(t)$ be identified with their limiting values λ_b^- and λ_d^- , then: $\eta(t)/\psi(t) \rightarrow 1/(1-\rho) = \lambda_b^-/(\lambda_b^- - \lambda_d^-)$ where $\rho := \lambda_d^-/\lambda_b^-$ (< 1 in the supercritical regime).

Thus, with $\kappa = 1/(1-\rho) > 1$

$$(7) \quad \mathbf{P}(\tau_e > t) \rightarrow \mathbf{P}(\tau_e = \infty) = \frac{p_0}{q\kappa + p_0p} \text{ as } t \rightarrow \infty.$$

The extinction probability reads

$$\rho_e = \mathbf{P}(\tau_e < \infty) = 1 - \mathbf{P}(\tau_e = \infty) = \frac{q(\kappa - p_0)}{q\kappa + p_0p} = \phi_0(\rho),$$

$N(t)$ vanishes at time $\tau_e \mid \tau_e < \infty$. Note $\rho_e \rightarrow 0$ as $x(0) \rightarrow \infty$ (or $q \rightarrow 0$) and $\phi_t(z) \rightarrow \rho_e = \phi_0(\rho)$ as $t \rightarrow \infty$.

We have

$$\tau_e = \inf(t > 0 : N_d(t) \geq N_b(t))$$

the first time the ever dead outnumber the ever living. In the randomized setup, even in the supercritical regime of mean exponential growth for $x(t) = x_b(t) - x_d(t)$, there is a “no-luck” possibility that this event occurs and this corresponds to global extinction of the population at time τ_e .

These results answer the somehow related question on if and when the ever dead ($N_d(t)$) outnumber the ever living ($N_b(t)$) in the supercritical linear birth and death Markov chain model; but so far the question on whether the ever dead ($N_d(t)$) outnumber the living ($N(t) = N_b(t) - N_d(t)$) has not been addressed.

3.2. The joint law of $N_b(t)$ and $N_d(t)$. In order to access to a comparative study between $N_d(t)$ and $N(t) := N_b(t) - N_d(t)$, we need to first compute the joint pgf of $N_b(t)$ and $N_d(t)$.

3.2.1. Joint pgf of $N_b(t)$ and $N_d(t)$. Define $N_b(t)$ and $N_d(t)$ as the number of people ever born and ever dead till time t . We wish now to compute the joint law of $N_b(t)$ and $N_d(t)$, keeping in mind $N(t) = N_b(t) - N_d(t)$ and for some $N(0) = N_b(0)$.

With $n = n_b - n_d$, the joint transition probabilities are obtained as

$$(8) \quad \mathbf{P}(N_b(t+dt) = n_b + 1, N_d(t+dt) = n_d \mid N(t) = n) = \lambda_b(t) n dt + o(dt)$$

$$\mathbf{P}(N_b(t+dt) = n_b, N_d(t+dt) = n_d + 1 \mid N(t) = n) = \lambda_d(t) n dt + o(dt)$$

allowing to derive the evolution equation of $\mathbf{EP}(N_b(t) = n_b, N_d(t) = n_d \mid N(0))$.

Introducing the joint pgf of $N_b(t)$ and $N_d(t)$ as $\Phi_t(z_b, z_d) := \mathbf{E}\left(z_b^{N_b(t)} z_d^{N_d(t)}\right)$, we obtain its time evolution as

$$(9) \quad \partial_t \Phi_t = (\lambda_b(t)(z_b - 1) + \lambda_d(t)(z_d - 1))(z_b \partial_{z_b} - z_d \partial_{z_d}) \Phi_t,$$

with initial condition $\Phi_0(z_b, z_d) = \phi_0(z_b)$, noting $N_d(0) = 0$.

3.2.2. Correlations. Consistently, we have $x_b(t) = \partial_{z_b} \Phi_t(1, 1)$ and $x_d(t) = \partial_{z_d} \Phi_t(1, 1)$. Taking the derivative of (9) with respect to z_b and also with respect to z_d and evaluating the results at $z_b = z_d = 1$ we are lead to $\dot{x}_b(t) = \lambda_b(t)(x_b(t) - x_d(t)) = \lambda_b(t)x(t)$, $x_b(0) = x(0)$ and to $\dot{x}_d(t) = \lambda_d(t)x(t)$, $x_d(0) = 0$. With

$$\mathbf{E}(N_b(t)^2) := \partial_{z_b z_b}^2 \Phi_t(1, 1), \quad \mathbf{E}(N_d(t)^2) := \partial_{z_d z_d}^2 \Phi_t(1, 1), \quad \mathbf{E}(N_b(t) N_d(t)) := \partial_{z_b z_d}^2 \Phi_t(1, 1)$$

we also have

$$\frac{d}{dt} \begin{bmatrix} \mathbf{E}(N_b(t)^2) \\ \mathbf{E}(N_d(t)^2) \\ \mathbf{E}(N_b(t) N_d(t)) \end{bmatrix} = \begin{bmatrix} 2\lambda_b(t) & 0 & -2\lambda_b(t) \\ 0 & -2\lambda_d(t) & 2\lambda_d(t) \\ \lambda_d(t) & -\lambda_b(t) & \lambda_b(t) - \lambda_d(t) \end{bmatrix} \begin{bmatrix} \mathbf{E}(N_b(t)^2) \\ \mathbf{E}(N_d(t)^2) \\ \mathbf{E}(N_b(t) N_d(t)) \end{bmatrix},$$

with initial condition $\begin{bmatrix} \mathbf{E}(N_b(0)^2) \\ \mathbf{E}(N_d(0)^2) \\ \mathbf{E}(N_b(0) N_d(0)) \end{bmatrix} = \begin{bmatrix} \mathbf{E}(N(0)^2) \\ 0 \\ 0 \end{bmatrix}$. In vector form,

this is also $\dot{\mathbf{x}}(t) = C(t)\mathbf{x}(t)$, $\mathbf{x}(0)$ where $\mathbf{x}(t) \succeq \mathbf{0}$ (componentwise).

The matrix $C(t)$ tends to a limit $C(\infty)$ as $t \rightarrow \infty$.

It has eigenvalues $(0, \lambda_b(t) - \lambda_d(t), 2(\lambda_b(t) - \lambda_d(t)))$, with

$$(0, \lambda_b(t) - \lambda_d(t), 2(\lambda_b(t) - \lambda_d(t))) \rightarrow (0, \lambda_b^- - \lambda_d^-, 2(\lambda_b^- - \lambda_d^-)),$$

as $t \rightarrow \infty$. We identify now $\lambda_1(t) = 2(\lambda_b(t) - \lambda_d(t))$ as its dominant eigenvalue (the one for which $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(s) ds$ is largest). Letting $X(t)$ be a 3×3 matrix with $X(0) = I$ and $\dot{X}(t) = C(t)X(t)$, we have $\mathbf{x}(t) = X(t)\mathbf{x}(0)$ and

$$e^{-\int_0^t \lambda_1(s) ds} X(t) \rightarrow \frac{\mathbf{x}\mathbf{y}'}{\mathbf{x}'\mathbf{y}} \text{ as } t \rightarrow \infty,$$

with $C(\infty)\mathbf{x} = \theta\mathbf{x}$, $\mathbf{y}'C(\infty) = \theta\mathbf{y}'$, $\theta = 2(\lambda_b^- - \lambda_d^-) := \lambda_1(\infty)$ the dominant eigenvalue of $C(\infty)$. The left and right eigenvectors of $C(\infty)$ associated to θ are found to be $\mathbf{x}' = ((\lambda_b^-)^2, (\lambda_d^-)^2, \lambda_b^- \lambda_d^-)$ and $\mathbf{y}' = (1, 1, -2)$. The latter convergence result holds because, with \mathbf{y}' independent of t , $\mathbf{y}'C(t) = \lambda_1(t)\mathbf{y}'$ for all $t > 0$ and $C(t)\mathbf{1} = \mathbf{0}$ ($\mathbf{1}$ is in the kernel of $C(t)$ for all t).

Indeed,

$$X(t) = \mathbf{x}(t)\mathbf{y}' \text{ solves } \dot{X}(t) = C(t)X(t)$$

and from Theorem 1 in [10], assuming⁴

$$\int_0^\infty |\dot{\lambda}_b(t)| dt < \infty \text{ and } \int_0^\infty |\dot{\lambda}_d(t)| dt < \infty,$$

then $\mathbf{x}(t) \simeq \mathbf{x} e^{\int_0^t \lambda_1(s) ds}$ (for large $t \gg T$).

We are thus led to

$$(10) \quad e^{-\int_0^t \lambda_1(s) ds} \text{Cov}(N_b(t), N_d(t)) \rightarrow C_{b,d} - C_b C_d > 0 \text{ as } t \rightarrow \infty$$

⁴The homographic birth and death rates satisfy these conditions. More generally, the conditions hold for any non increasing rate function λ of the form $\lambda(t) = a + (b - a)\mathbf{P}(T > t)$ where $b > a > 0$ and $\mathbf{P}(T > t)$ is the tail probability distribution of any positive random variable T .

where $C_{b,d} = \mathbf{E} \left(N_b(0)^2 \right) (\lambda_b^- \lambda_d^-) / (\lambda_b^- - \lambda_d^-)^2$, $C_b = \mathbf{E} (N_b(0)) \lambda_b^- / (\lambda_b^- - \lambda_d^-)$ and $C_d = \mathbf{E} (N_b(0)) \lambda_d^- / (\lambda_b^- - \lambda_d^-)$ so $C_{b,d} - C_b C_d = \text{Var}(N_b(0)) (\lambda_b^- \lambda_d^-) / (\lambda_b^- - \lambda_d^-)^2 > 0$, translating an asymptotic positive correlation of $(N_b(t), N_d(t))$. Note

$$\frac{1}{t} \int_0^t \lambda_1(s) ds \rightarrow \theta \text{ as } t \rightarrow \infty.$$

3.2.3. Joint pgf revisited. Introducing the change of variables from (z_b, z_d) to $\xi \equiv \sqrt{z_b/z_d}$ and $\zeta \equiv \sqrt{z_b z_d}$, the right-hand-side of (9) only contains ∂_ξ . The evolution equation for $\tilde{\Phi}$ in the new variables indeed reads

$$\partial_t \tilde{\Phi}_t = (\lambda_b(t) \zeta \xi^2 - (\lambda_b(t) + \lambda_d(t)) \xi + \lambda_d(t) \zeta) \partial_\xi \tilde{\Phi}_t, \quad \tilde{\Phi}_0(\xi, \zeta) = \phi_0(\xi \zeta).$$

$\tilde{\Phi}_t$ is therefore obtained as a function of a suitable combination of ξ and time (solvable by the method of characteristics), and separately of ζ in a way determined by initial conditions.

Fixing ζ to a certain value, one may then seek for an homographic ansatz

$$(11) \quad \tilde{\Phi}_t(\xi, \zeta) = \tilde{\Phi}_0 \left(\frac{a(t)\xi + b(t)}{c(t)\xi + d(t)}, \zeta \right) = \phi_0 \left(\frac{a(t)\xi + b(t)}{c(t)\xi + d(t)} \zeta \right),$$

with $a(0) = d(0) = 1$ and $b(0) = c(0) = 0$.

Plugging this ansatz into the evolution equation for $\tilde{\Phi}$ and denoting respectively

$$\alpha(t) = \lambda_b(t) \zeta, \quad \beta(t) = -(\lambda_b(t) + \lambda_d(t)), \quad \gamma(t) = \lambda_d(t) \zeta,$$

we need to solve $\dot{\xi}(t) = \alpha(t) \xi(t)^2 + \beta(t) \xi(t) + \gamma(t)$ with initial condition $\xi(0) = \xi$ and the guess $\xi(t) = \frac{a(t)\xi + b(t)}{c(t)\xi + d(t)}$. We get

$$\begin{aligned} \dot{a}c - a\dot{c} &= (ad - bc)\alpha \\ \dot{b}c - b\dot{c} + \dot{a}d - a\dot{d} &= (ad - bc)\beta \\ \dot{b}d - b\dot{d} &= (ad - bc)\gamma. \end{aligned}$$

Fixing the CP^3 gauge choice of any homographic form $\frac{a\xi + b}{c\xi + d}$ (invariant under $a, b, c, d \rightarrow \lambda a, \lambda b, \lambda c, \lambda d$) by fixing $ad - bc = 1$ yields

$$(12) \quad \frac{d}{dt} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \beta/2 & -\alpha \\ \gamma & -\beta/2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

to be solved as the ordered exp-integral

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} (t) = \overleftarrow{\exp} \int_0^t \begin{bmatrix} \beta/2 & -\alpha \\ \gamma & -\beta/2 \end{bmatrix} (s) ds.$$

Although this system can rarely be solved explicitly by quadrature for all t , this in principle solves $\tilde{\Phi}_t(\xi, \zeta)$ hence $\Phi_t(z_b, z_d)$ when back to the original variables.

Remark: We note from the above differential equation (12) and the expression of (α, β, γ) in the transition matrix that, as functions of ζ

$$(a, b/\zeta, c/\zeta, d) \text{ are functions of } \zeta^2,$$

or else that (besides t)

$$(13) \quad \zeta a/c \text{ and } \zeta d/b \text{ are functions of } \zeta^2.$$

3.2.4. Asymptotics. In matrix form, (12) reads $\dot{X}(t) = A(t)X(t)$, $X(0) = I$ where $A(t) = \begin{bmatrix} \beta/2 & -\alpha \\ \gamma & -\beta/2 \end{bmatrix}$ has zero trace and $X(t) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}(t)$. It follows that $\det(X)$ is a conserved quantity.

Note that, due to $|\zeta| \leq 1$, $\det(A(t)) = -(\lambda_b(t) + \lambda_d(t))^2/4 + \zeta^2 \lambda_b(t) \lambda_d(t) \leq -(\lambda_b(t) - \lambda_d(t))^2/4 < 0$: For all t , the eigenvalues of $A(t)$ are real with opposite sign $\mu_{\pm}(t) = \pm \frac{1}{2} \sqrt{(\lambda_b(t) + \lambda_d(t))^2 - 4\zeta^2 \lambda_b(t) \lambda_d(t)}$. With

$$A_1 = \begin{bmatrix} -1/2 & -\zeta \\ 0 & 1/2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -1/2 & 0 \\ \zeta & 1/2 \end{bmatrix},$$

we have $A(t) = \lambda_b(t) A_1 + \lambda_d(t) A_2$ with $[A_1, A_2] \neq 0$; In fact A_1, A_2 generate the full $su(2)$ algebra hence it is consistent to define

$$(14) \quad X(t) = \exp(\alpha_1(t) L_1 + \alpha_2(t) L_2 + \alpha_3(t) L_3),$$

for some $\alpha_i(t)$ obeying a Wei-Norman type non-linear differential equation [8] and L_i Pauli matrices. For recent further developments, see [12].

With $i = 1, 2$, let $\mathbf{x}_i(t) = X(t) \mathbf{x}_i(0)$ with $\mathbf{x}_1(0)' = (1, 0) \Rightarrow \mathbf{x}_1(t)' = (a, b)$ and $\mathbf{x}_2(0)' = (0, 1) \Rightarrow \mathbf{x}_2(t)' = (c, d)$.

We set

$$\begin{aligned} A(t) &= A(\infty) + \tilde{A}(t) \\ A(\infty) &= \lambda_b^- A_1 + \lambda_d^- A_2 \\ \tilde{A}(t) &= (\lambda_b(t) - \lambda_b^-) A_1 + (\lambda_d(t) - \lambda_d^-) A_2. \end{aligned}$$

Of course, the matrix $A(\infty)$ has two real opposite eigenvalues

$$\mu_{\pm} = \pm \frac{1}{2} \sqrt{(\lambda_b^- + \lambda_d^-)^2 - 4\zeta^2 \lambda_b^- \lambda_d^-}$$

with $\mu_{\pm}(t) \rightarrow \mu_{\pm}$ as $t \rightarrow \infty$.

From Theorem 1 in [10], assuming $\int^{\infty} |\dot{\lambda}_b(t)| dt < \infty$ and $\int^{\infty} |\dot{\lambda}_d(t)| dt < \infty$, with T the (large) time after which the process enters its asymptotic regime, we have

$$(15) \quad \mathbf{x}_i(t) \sim \exp\left(\int_T^t \mu_+(s) ds\right) \begin{bmatrix} \phi_+(1) & \phi_+(1) \\ \phi_+(2) & \phi_+(2) \end{bmatrix} \mathbf{x}_i(T), \text{ as } t \rightarrow \infty$$

where ϕ_+ is the right column eigenvector of $A(\infty)$ associated to μ_+ , i.e. $A(\infty) \phi_+ = \mu_+ \phi_+$. The row vector ϕ'_+ is

$$\phi'_+ := (\phi_+(1), \phi_+(2)) = (1, (\mu_+ + (\lambda_b^- + \lambda_d^-)/2) / (-\zeta \lambda_b^-)).$$

We conclude that, as $t \rightarrow \infty$, both $|\mathbf{x}_i(t)| \rightarrow \infty$. Recalling $\det(X) = a(t)d(t) - b(t)c(t) = |\mathbf{x}_1(t)| |\mathbf{x}_2(t)| \sin(\widehat{\mathbf{x}_1(t), \mathbf{x}_2(t)}) = 1$ for all t , it follows that the angle $\widehat{\mathbf{x}_1(t), \mathbf{x}_2(t)}$ goes to 0 as $t \rightarrow \infty$ at rate $2 \int_T^t \mu_+(s) ds > 0$: the vectors $\mathbf{x}_1(t), \mathbf{x}_2(t)$ become parallel (or anti-parallel) with the same time evolution $\sim \exp\left(\int_T^t \mu_+(s) ds\right)$ which guarantees that they stay homothetical at large $t \gg T$ (as seen from the asymptotic evaluation (15)). Therefore both $a(t)/c(t)$ and $b(t)/d(t)$ tend to the same limit (homothetical ratio $\mathbf{x}_1/\mathbf{x}_2$) as $t \rightarrow \infty$, namely

$(a(T) + b(T)) / (c(T) + d(T))$. This common limit exists and, from the above scaling argument (13), it can be written in the form say $h(\zeta^2) / \zeta$, for some (unknown) function h encoding the finite size effects till time T ; it depends on ζ and ζ^2 . Note that, consistently, both $a(t) / b(t)$ and $c(t) / d(t)$ also have a common limit (angular azimuth of both $\mathbf{x}_1, \mathbf{x}_2$) which is

$$\phi_+(1) / \phi_+(2) = (-\zeta \lambda_b^-) / (\mu_+ + (\lambda_b^- + \lambda_d^-) / 2) = u(\zeta^2) / \zeta,$$

where $u(\zeta^2) = (2\mu_+ - (\lambda_b^- + \lambda_d^-)) / (2\lambda_d^-)$.

So the limit structure of $X(t)$ is known and as $t \rightarrow \infty$

$$(16) \quad \tilde{\Phi}_t(\xi, \zeta) \rightarrow \phi_0(h(\zeta^2)).$$

The dependence on ξ has disappeared in the limit. Coming back to the original variables, recalling $\xi \equiv \sqrt{z_b/z_d}$ and $\zeta \equiv \sqrt{z_b z_d}$, as $t \rightarrow \infty$ yields the limiting shape of $\Phi_t(z_b, z_d)$. We get

$$\Phi_t(z_b, z_d) = \mathbf{E} \left(z_b^{N_b(t)} z_d^{N_d(t)} \right) \rightarrow_{t \rightarrow \infty} \mathbf{E} \left(z_b^{N_b(\infty)} z_d^{N_d(\infty)} \right) = \phi_0(h(z_b z_d)).$$

3.2.5. A particular solvable case. It turns out that in several case there is an explicit solution to (12), illustrating the general conclusions just discussed. Let us give an example.

• **Explicit solution.** Let $\rho > 0$ be a fixed number. When $\lambda_d(t) = \rho \lambda_b(t)$, (12) can be solved explicitly while observing

$$\begin{bmatrix} \beta/2 & -\alpha \\ \gamma & -\beta/2 \end{bmatrix} = \lambda_b(t) \begin{bmatrix} -\frac{(1+\rho)}{2} & -\zeta \\ \zeta\rho & \frac{(1+\rho)}{2} \end{bmatrix}$$

and diagonalizing the latter matrix. With $\Lambda_b(t) = \int_0^t \lambda_b(s) ds$ and putting

$$\theta(\zeta^2) = \frac{1}{2} \sqrt{(1+\rho)^2 - 4\rho\zeta^2},$$

we indeed get $ad - bc = 1$ where

$$\begin{aligned} a &= -\frac{(1+\rho)}{2\theta(\zeta^2)} \sinh(\theta(\zeta^2) \Lambda_b(t)) + \cosh(\theta(\zeta^2) \Lambda_b(t)) \\ b &= \frac{\zeta\rho}{\theta(\zeta^2)} \sinh(\theta(\zeta^2) \Lambda_b(t)) \\ c &= -\frac{\zeta}{\theta(\zeta^2)} \sinh(\theta(\zeta^2) \Lambda_b(t)) \\ d &= \frac{(1+\rho)}{2\theta(\zeta^2)} \sinh(\theta(\zeta^2) \Lambda_b(t)) + \cosh(\theta(\zeta^2) \Lambda_b(t)). \end{aligned}$$

so that

$$\tilde{\Phi}_t(\xi, \zeta) = \tilde{\Phi}_0 \left(\frac{a\xi + b}{c\xi + d}, \zeta \right) = \phi_0 \left(\frac{a\xi + b}{c\xi + d} \zeta \right).$$

Recalling $\xi\zeta = z_b$ and $\zeta^2 = z_b z_d$, coming back to the original variables, we get

$$\Phi_t(z_b, z_d) = \mathbf{E} \left(z_b^{N_b(t)} z_d^{N_d(t)} \right) = \phi_0 \left(\frac{A_t(z_b z_d) z_b + B_t(z_b z_d) z_b z_d}{C_t(z_b z_d) z_b + D_t(z_b z_d)} \right)$$

where

$$\begin{aligned} A_t(z_b z_d) &= -\frac{1+\rho}{2\theta(z_b z_d)} \sinh(\theta(z_b z_d) \Lambda_b(t)) + \cosh(\theta(z_b z_d) \Lambda_b(t)) \\ B_t(z_b z_d) &= \frac{\rho}{\theta(z_b z_d)} \sinh(\theta(z_b z_d) \Lambda_b(t)) \\ C_t(z_b z_d) &= -\frac{1}{\theta(z_b z_d)} \sinh(\theta(z_b z_d) \Lambda_b(t)) \\ D_t(z_b z_d) &= \frac{1+\rho}{2\theta(z_b z_d)} \sinh(\theta(z_b z_d) \Lambda_b(t)) + \cosh(\theta(z_b z_d) \Lambda_b(t)) \end{aligned}$$

each depend on t and $z_b z_d$.

• $t \rightarrow \infty$ **asymptotics.**

This fully solvable case enables us to obtain exact asymptotics for the generating function $\Phi_t(z_b, z_d)$ in the supercritical case $\rho < 1$. From the explicit formulae for (a, b, c, d) we get:

$$\lim_{t \rightarrow \infty} \frac{a(t)}{c(t)} = \frac{1+\rho-2\theta(\zeta^2)}{2\zeta} = \frac{2\zeta\rho}{1+\rho+2\theta(\zeta^2)} = \lim_{t \rightarrow \infty} \frac{b(t)}{d(t)}$$

identified indeed by definition of $\theta(\zeta^2)$. This illustrates the fact that the vectors $\mathbf{x}_i(t)$, $i = 1, 2$ become parallel up to $\exp(-2\Lambda_b(t))$ when t gets large. Note that one still have of course $ad - bc = 1 \Leftrightarrow (ad)/(bc) - 1 = 1/(bc) = O(\exp(-2\Lambda_b(t)))$. Indeed, it appears that $a/c - b/d = 1/(cd) = O(\exp(-2\Lambda_b(t))) \rightarrow 0$ (without being strictly 0). Hence we get that the generating function yields

$$\Phi_t(z_b, z_d) \rightarrow \phi_0\left(\frac{1+\rho-2\theta}{2}\right), \quad \theta = \frac{1}{2}\sqrt{(1+\rho)^2 - 4\rho z_b z_d}.$$

Dependence on ξ has disappeared. Observing that as $t \rightarrow \infty$

$$(17) \quad \Phi_t(z_b, z_d) = \mathbf{E}\left(z_b^{N_b(t)} z_d^{N_d(t)}\right) \rightarrow \rho_e \mathbf{E}\left((z_b z_d)^{N_d(\infty)} \mid \text{extinct}\right),$$

because either extinction occurred (with probability ρ_e) and $N_b(\infty) = N_d(\infty) < \infty$ or not in which case $N_b(\infty) = N_d(\infty) = \infty$, we conclude

$$(18) \quad \rho_e \mathbf{E}\left(z^{N_d(\infty)} \mid \text{extinct}\right) = \phi_0\left(\frac{1+\rho-\sqrt{(1+\rho)^2 - 4\rho z}}{2}\right).$$

Recalling $\rho_e = \phi_0(\rho)$, we obtain the pgf of the cumulative population size (ever born or ever dead) $N_b(\infty) = N_d(\infty) < \infty$, given extinction occurred, as the compound expression

$$\begin{aligned} \mathbf{E}\left(z^{N_d(\infty)} \mid \text{extinct}\right) &= \phi_0(h(z))/\phi_0(h(1)) \\ \text{where } h(z) &= \frac{1+\rho-\sqrt{(1+\rho)^2 - 4\rho z}}{2} \end{aligned}$$

In particular (with $h(1) = \rho = \lambda_d^-/\lambda_b^- < 1$)

$$\mathbf{E}(N_d(\infty) \mid \text{extinct}) = \frac{\phi_0'(h(1))}{\phi_0(h(1))} h'(1) = \frac{\phi_0'(\rho)}{\phi_0(\rho)} \frac{\rho}{1-\rho}.$$

Note that, would one consider a subcritical case $\rho > 1$, then $h(1) = 1$ leading to $\rho_e = \phi_0(h(1)) = 1$ (almost sure extinction) with $\mathbf{E}(z^{N_d(\infty)}) = \phi_0(h(z))$ characterizing the law of $N_b(\infty) = N_d(\infty) < \infty$ at extinction. Note $\mathbf{E}(N_d(\infty)) = \mathbf{E}(N_b(0)) \frac{\rho}{\rho-1}$.

In the critical case $\rho = 1$, $h(1) = 1$ ($\rho_e = 1$) and $\mathbf{E}(z^{N_d(\infty)}) = \phi_0(1 - \sqrt{1-z})$ with $\mathbf{E}(N_d(\infty)) = \infty$.

Remark: This solvable case includes the situation where both $\lambda_b(t)$ and $\lambda_d(t)$ are independent of time. It suffices to particularize the latter formula while setting $\Lambda_b(t) = \lambda_b t$. \square

If $z_b = z$, $z_d = z^{-1}$,

$$\Phi_t(z, z^{-1}) = \mathbf{E}(z^{N(t)}) = \phi_0\left(\frac{A_t(1)z + B_t(1)}{C_t(1)z + D_t(1)}\right)$$

which is the previous homographic expression for $\mathbf{E}(z^{N(t)})$ in the particular case when $\lambda_d(t) = \rho\lambda_b(t)$.

If $z_b = z$, $z_d = 1$ or $z_b = 1$, $z_d = z$, we get the marginals

$$\begin{aligned}\Phi_t(z, 1) &= \mathbf{E}(z^{N_b(t)}) = \phi_0\left(z \frac{A_t(z) + B_t(z)}{C_t(z)z + D_t(z)}\right) \text{ or} \\ \Phi_t(1, z) &= \mathbf{E}(z^{N_d(t)}) = \phi_0\left(\frac{A_t(z) + B_t(z)z}{C_t(z) + D_t(z)}\right).\end{aligned}$$

The processes $(N_b(t), N_d(t))$ both are monotone non-decreasing.

If $z_b = z^{-1}$, $z_d = z^2$, we are interested in the Laurent series

$$(19) \quad \psi_t(z) := \mathbf{E}(z^{2N_d(t) - N_b(t)}) = \Phi_t(z^{-1}, z^2) = \phi_0\left(\frac{A_t(z)z^{-1} + B_t(z)z}{C_t(z)z^{-1} + D_t(z)}\right).$$

The quantity $\psi_t(z) = \Phi_t(z^{-1}, z^2)$ turns out to be useful in the understanding of the statistical properties of the first time that $\Delta(t) := 2N_d(t) - N_b(t)$ hits 0, starting from $\Delta(0) = -N_b(0) = -N(0) < 0$. We now address this point.

3.2.6. First time $N_d(t) \geq N(t) = N_b(t) - N_d(t)$ ($\Delta(t) \geq 0$). The process $\Delta(t)$ takes values in \mathbb{Z} . It moves up by 2 units when a death event occurs and down by 1 unit when a birth event occurs. Starting from $\Delta(0) = -N_b(0) < 0$, the process $\Delta(t)$ will enter the nonnegative region $\Delta(t) \geq 0$ for the first time at some random crossing time $\tau_{\Delta(0)}$ where the dependence on the initial condition has been emphasized.

There are two types of crossing events:

- type-1: $\Delta(t)$ enters the nonnegative region $\Delta(t) \geq 0$ as a result of $\Delta(t) = -1$ followed instantaneously by a death event leading to $\Delta(t_+) = +1$.
- type-2: $\Delta(t)$ enters the nonnegative region $\Delta(t) \geq 0$ as a result of $\Delta(t) = -2$ followed instantaneously by a death event leading to $\Delta(t_+) = 0$.

Consider the process $\Delta_1(t)$ which is $\Delta(t)$ conditioned on $t = 0$ being a type-1 first crossing time ($\Delta(0) = -1$ and $\Delta(0_+) = +1$). Let $\lambda_1(t)$ be the rate at which some

type-1 crossing occurs at t for Δ_1 . We let

$$\widehat{\lambda}_1(s) := \int_0^\infty e^{-st} \lambda_1(t) dt$$

be the Laplace-Stieltjes transform (LST) of $\lambda_1(t)$.

Let τ_1 denote the (random) time between 2 consecutive type-1 crossing times for Δ_1 . By standard renewal arguments [5]

$$(20) \quad \widehat{\lambda}_1(s) = 1 / \left(1 - \widehat{f}_1(s) \right),$$

where $\widehat{f}_1(s)$ is the LST of the law of τ_1 .

Similarly, consider the process $\Delta_2(t)$ which is $\Delta(t)$ conditioned on $t = 0$ being a type-2 first crossing time ($\Delta(0) = -2$ and $\Delta(0_+) = 0$). Let $\lambda_2(t)$ be the rate at which some type-2 crossing occurs at t for Δ_2 . Let τ_2 denote the random time between 2 consecutive type-2 crossing times for Δ_2 . Then $\widehat{\lambda}_2(s) = 1 / \left(1 - \widehat{f}_2(s) \right)$ where $\widehat{\lambda}_1(s)$ is the LST of $\lambda_2(t)$ and $\widehat{f}_2(s)$ the LST of the law of τ_2 .

Clearly, between any two consecutive crossings of any type, there can be none, one or more crossings of the other type.

The laws of the first return times (τ_1, τ_2) of (Δ_1, Δ_2) are thus known once the rates $(\lambda_1(t), \lambda_2(t))$ are.

Let now $\lambda_{1,\Delta(0)}(t)$ (respectively $\lambda_{2,\Delta(0)}(t)$) be the rate at which some type-1 (respectively type-2) crossing occurs at t when Δ is now simply conditioned on any $\Delta(0) < 0$.

Let also $\tau_{1,\Delta(0)}$ (respectively $\tau_{2,\Delta(0)}$) be the first time that some type-1 (respectively type-2) crossing occurs for Δ given $\Delta(0) < 0$.

Again by renewal arguments for this now delayed renewal process:

$$(21) \quad \widehat{\lambda}_{1,\Delta(0)}(s) = \widehat{f}_{1,\Delta(0)}(s) / \left(1 - \widehat{f}_1(s) \right) \text{ and}$$

$$\widehat{\lambda}_{2,\Delta(0)}(s) = \widehat{f}_{2,\Delta(0)}(s) / \left(1 - \widehat{f}_2(s) \right)$$

where $\widehat{\lambda}_{1,\Delta(0)}(s)$ (respectively $\widehat{\lambda}_{2,\Delta(0)}(s)$) is the LST of $\lambda_{1,\Delta(0)}(t)$ (respectively $\lambda_{2,\Delta(0)}(t)$) and $\widehat{f}_{1,\Delta(0)}(s)$ (respectively $\widehat{f}_{2,\Delta(0)}(s)$) is the LST of the law of $\tau_{1,\Delta(0)}$ (respectively $\tau_{2,\Delta(0)}$). Finally, we clearly have

$$(22) \quad \tau_{\Delta(0)} = \inf(\tau_{1,\Delta(0)}, \tau_{2,\Delta(0)}).$$

The knowledge of all these quantities (in particular the law of $\tau_{\Delta(0)}$) is therefore conditioned on the knowledge of the rates. Conditioned on the initial condition $\Delta(0) < 0$, we have for instance

$$(23) \quad \lambda_{1,\Delta(0)}(t) dt = [z^{-1}] \psi_t(z) \cdot \lambda_d(t) \cdot \mathbf{E}_{\Delta(0)}(N_d(t) + 1 \mid \Delta(t) = -1) dt$$

$$\lambda_{2,\Delta(0)}(t) dt = [z^{-2}] \psi_t(z) \cdot \lambda_d(t) \cdot \mathbf{E}_{\Delta(0)}(N_d(t) + 2 \mid \Delta(t) = -2) dt,$$

corresponding respectively, through a jump of size 2 of $\Delta(t)$, to the death events $\Delta(t+dt) = 1$ given $\Delta(t) = -1$ and $\Delta(t+dt) = 0$ given $\Delta(t) = -2$. In (23), $\psi_t(z) = \Phi_t(z^{-1}, z^2)$ is given by (19).

The term $[z^{-1}] \psi_t(z)$ (respectively $[z^{-2}] \psi_t(z)$) is the probability that $\Delta(t) = -1$ (respectively $\Delta(t) = -2$).

The term $\mathbf{E}(N_d(t) + 1 \mid \Delta(t) = -1)$ (respectively $\mathbf{E}(N_d(t) + 2 \mid \Delta(t) = -2)$) is the expected value of $N(t) = N_b(t) - N_d(t)$ given $\Delta(t) = -1$ (respectively of $N(t)$ given $\Delta(t) = -2$).

These last two informations can be extracted from the joint probability generating function of $(N_d(t), \Delta(t) := 2N_d(t) - N_b(t) \mid \Delta(0))$ which is known from $\Phi_t(z_b, z_d) = \mathbf{E}\left(z_b^{N_b(t)} z_d^{N_d(t)}\right)$.

Therefore, the law of $\tau_{\Delta(0)}$ follows in principle, although the actual computational task left remains huge.

4. AGE-STRUCTURED MODELS

We finally address a different point of view of the birth/death problems, once again deterministic but now based on age-structured models (see e.g. [4], [3], [6] and [13]).

Let $\rho = \rho(a, t)$ be the density of individuals of age a at time t . Then ρ obeys

$$(24) \quad \partial_t \rho + \partial_a \rho = -\lambda_d(a, t) \rho,$$

with boundary conditions

$$\rho(a, 0) = \rho_0(a) \text{ and } \rho(0, t) = \int \lambda_b(a, t) \rho(a, t) da.$$

The quantities $\lambda_b(a, t)$ and $\lambda_d(a, t)$ are the birth and death rates at age and time (a, t) and they are the inputs of the model together with $\rho_0(a)$. The equation (24) with its boundary conditions constitute the Lotka-McKendrick-Von Foerster model, [11].

The total population mean size at t is

$$x(t) = \int \rho(a, t) da.$$

With $\dot{\bullet} \equiv d/dt$, integrating (24) with respect to age and observing $\rho(\infty, t) = 0$, we get

$$\dot{x}(t) = \int \lambda_b(a, t) \rho(a, t) da - \int \lambda_d(a, t) \rho(a, t) da =: \dot{x}_b(t) - \dot{x}_d(t)$$

where $\dot{x}_b(t)$ and $\dot{x}_d(t)$ are the birth and death rates at time t . We note that $\dot{x}_b(t) = \rho(0, t)$, the rate at which new individuals (of age $a = 0$) are injected in the system.

Integrating, we have $x(t) = x_b(t) - x_d(t)$, with $x(0) = x_b(0)$ in view of $\lambda_d(a, 0) = 0$. $x_b(t)$ and $x_d(t)$ are the mean number of individuals ever born and ever dead between times 0 and t ($x_b(t)$ including, as before, the initial individuals). The answer to the question of whether and when the ever dead outnumbered the living is transferred to the existence of the time t_* when some overshooting occurred, defined by $\frac{x_d(t_*)}{x(t_*)} = 1$. As in Section 2, this requires the computation of both $x_d(t)$ and $x(t)$ and we now address this question.

With $a \wedge t = \min \{a, t\}$, let

$$(25) \quad \bar{\Lambda}_d(a, t) = \int_0^{a \wedge t} \lambda_d(a-s, t-s) ds.$$

Integration of (24), using the method of characteristics yields its solution as

$$(26) \quad \begin{aligned} \rho(a, t) &= \rho_0(a-t) e^{-\bar{\Lambda}^d(a, t)} \text{ if } t < a \\ &= \dot{x}_b(t-a) e^{-\bar{\Lambda}^d(a, t)} \text{ if } t > a. \end{aligned}$$

Recall $\rho_0(a) = \rho(a, 0)$ and $\dot{x}_b(t) = \rho(0, t)$ as from the boundary conditions of (24). We shall distinguish 3 cases.

4.1. Case: $\lambda_b(a, t)$ and $\lambda_d(a, t)$ independent of age a ⁵. Suppose $\lambda_b(a, t) = \lambda_b(t)$ and $\lambda_d(a, t) = \lambda_d(t)$ for some given time-dependent rate functions $\lambda_b(t), \lambda_d(t)$. Then $\dot{x}_b(t) = \lambda_b(t)x(t)$ and $\dot{x}_d(t) = \lambda_d(t)x(t)$. Supposing $x(0) = \int \rho_0(a) da < \infty$, we get

$$x(t) = x(0) e^{\Lambda_b(t) - \Lambda_d(t)},$$

where Λ_b, Λ_d are the primitives of λ_b, λ_d . Furthermore,

$$\begin{aligned} \bar{\Lambda}_d(a, t) &= \int_0^{a \wedge t} \lambda_d(t-s) ds \\ &= \Lambda_d(t) \text{ if } t < a \\ &= \Lambda_d(t) - \Lambda_d(t-a) \text{ if } t > a. \end{aligned}$$

We thus obtain

$$\begin{aligned} x_b(t) &= x(0) + \int_0^t \lambda_b(s) x(s) ds = x(0) \left(1 + \int_0^t \lambda_b(s) e^{\Lambda_b(s) - \Lambda_d(s)} ds \right) \\ x_d(t) &= \int_0^t \lambda_d(s) x(s) ds = x(0) \int_0^t \lambda_d(s) e^{\Lambda_b(s) - \Lambda_d(s)} ds \end{aligned}$$

and for the age-structure

$$\begin{aligned} \rho(a, t) &= \rho_0(a-t) e^{-\Lambda_d(t)} \text{ if } t < a \\ &= \dot{x}_b(t-a) e^{-(\Lambda_d(t) - \Lambda_b(t-a))} \text{ if } t > a, \end{aligned}$$

which is completely determined because $\dot{x}_b(t) = x(0) \lambda_d(t) e^{\Lambda_b(t) - \Lambda_d(t)}$ is.

4.2. Case: $\lambda_b(a, t)$ and $\lambda_d(a, t)$ independent of time t ⁶. Suppose $\lambda_b(a, t) = \lambda_b(a)$ and $\lambda_d(a, t) = \lambda_d(a)$ for some given age-dependent rate functions $\lambda_b(a), \lambda_d(a)$. Then

$$\begin{aligned} \bar{\Lambda}_d(a, t) &= \int_0^{a \wedge t} \lambda_d(a-s) ds \\ &= \Lambda_d(a) - \Lambda_d(a-t) \text{ if } t < a \\ &= \Lambda_d(a) \text{ if } t > a. \end{aligned}$$

⁵Not very realistic since it assumes that individuals may reproduce at any age and are potentially immortal eg no cutoff to $\lambda_b(a)$ for $a < a_{crit}$ or $a > a_{crit}$, and $\lambda_d(a)$ does not increase with a .

⁶which amounts to assuming that no modification of the demographical dynamics ever occurs.

We thus obtain

$$\begin{aligned}\rho(a, t) &= \rho_0(a - t) e^{-(\Lambda_d(a) - \Lambda_d(a-t))} \text{ if } t < a \\ &= \dot{x}_b(t - a) e^{-\Lambda_d(a)} \text{ if } t > a\end{aligned}$$

which is completely determined as well once $\dot{x}_b(t) = \rho(0, t)$ is. We have

$$\begin{aligned}\dot{x}_b(t) &= \int \lambda_b(a) \rho(a, t) da \\ &= \int_0^t \lambda_b(a) \dot{x}_b(t - a) e^{-\Lambda_d(a)} da + \int_t^\infty \lambda_b(a) \rho_0(a - t) e^{-(\Lambda_d(a) - \Lambda_d(a-t))} da,\end{aligned}$$

the sum of two convolution terms. Taking the Laplace transforms, with

$$\widehat{\alpha}(z) := \int_0^\infty e^{-za} \lambda_b(a) e^{-\Lambda_d(a)} da \text{ and } \widehat{\beta}(z) := \int_0^\infty e^{-za} \rho_0(a) e^{\Lambda_d(a)} da,$$

we get the Laplace-transform of $\dot{x}_b(t)$ as

$$\widehat{x}_b(z) = \widehat{x}_b(z) \widehat{\alpha}(z) + \widehat{\beta}(z) \widehat{\alpha}(z),$$

leading to

$$\widehat{x}_b(z) = \frac{\widehat{\alpha}(z) \widehat{\beta}(z)}{1 - \widehat{\alpha}(z)}.$$

Inverting the Laplace transform yields $\dot{x}_b(t)$. The solution of (24) is completely determined from (26).

We also have

$$x(t) = \int \rho(a, t) da = \int_0^t \dot{x}_b(t - a) e^{-\Lambda_d(a)} da + \int_t^\infty \rho_0(a - t) e^{-(\Lambda_d(a) - \Lambda_d(a-t))} da$$

the sum of two convolution terms. Taking the Laplace transforms, with

$$\widehat{\alpha}_0(z) := \int_0^\infty e^{-za} e^{-\Lambda_d(a)} da,$$

we get

$$\widehat{x}(z) = \widehat{x}_b(z) \widehat{\alpha}_0(z) + \widehat{\beta}(z) \widehat{\alpha}_0(z).$$

So

$$(27) \quad \widehat{x}(z) = \frac{\widehat{\alpha}_0(z) \widehat{\beta}(z)}{1 - \widehat{\alpha}(z)} = \frac{\widehat{\alpha}_0(z)}{\widehat{\alpha}(z)} \widehat{x}_b(z).$$

Inverting this Laplace transform yields $x(t)$.

4.3. The complete case. This is the realistic case when both $\lambda_b(a, t)$ and $\lambda_d(a, t)$ fully depend on (a, t) . Looking at

$$\dot{x}(t) = \int \lambda_b(a, t) \rho(a, t) da - \int \lambda_d(a, t) \rho(a, t) da =: \dot{x}_b(t) - \dot{x}_d(t)$$

suggests to introduce the mean birth and death rates (averaging over age)

$$\lambda_b^*(t) = \frac{\int \lambda_b(a, t) \rho(a, t) da}{\int \rho(a, t) da} \text{ and } \lambda_d^*(t) = \frac{\int \lambda_d(a, t) \rho(a, t) da}{\int \rho(a, t) da}.$$

Then, with $x(t) = \int \rho(a, t) da$

$$\dot{x}(t) = (\lambda_b^*(t) - \lambda_d^*(t)) x(t) =: \dot{x}_b(t) - \dot{x}_d(t).$$

So

$$x(t) = x(0) e^{\Lambda_b^*(t) - \Lambda_d^*(t)},$$

where Λ_b^*, Λ_d^* are the primitives of λ_b^*, λ_d^* and

$$\dot{x}_b(t) = x(0) \lambda_b^*(t) e^{\Lambda_b^*(t) - \Lambda_d^*(t)}, \quad \dot{x}_d(t) = x(0) \lambda_d^*(t) e^{\Lambda_b^*(t) - \Lambda_d^*(t)}.$$

We are back to a case where λ_b^*, λ_d^* are independent of age although we know that there is an age dependency for both $\lambda_b(a, t)$ and $\lambda_d(a, t)$.

Suppose we are given $(\lambda_b^*(t), \lambda_d^*(t))$ in the first place. Then $\dot{x}_b(t), \dot{x}_d(t)$ and $x(t)$ are known from Subsection 4.1. Furthermore $\rho(a, t) = \rho^*(a, t)$ with

$$\begin{aligned} \rho^*(a, t) &= \rho_0(a - t) e^{-\Lambda_d^*(t)} \text{ if } t < a \\ &= \dot{x}_b(t - a) e^{-(\Lambda_d^*(t) - \Lambda_b^*(t - a))} \text{ if } t > a \end{aligned}$$

is the known age-dependent density.

So we need to solve the following inverse problem: find the unknown functions $\lambda_b(a, t)$ and $\lambda_d(a, t)$ such that

$$\int \lambda_b(a, t) \rho^*(a, t) da = \dot{x}_b(t) \text{ and } \int \lambda_d(a, t) \rho^*(a, t) da = \dot{x}_d(t)$$

which is a linear problem (not in the convolution class). Then $(x_b(t), x_d(t), x(t))$ will constitute a solution consistent with (24) and the rates $\lambda_b(a, t), \lambda_d(a, t)$.

We briefly sketch another point of view:

Suppose we were given $\lambda_d(a, t)$ and $\dot{x}_b(t) = \rho(0, t)$ in the first place.

Then $\rho(a, t)$ is completely known from Equations (25) and (26), together with $x(t) = \int \rho(a, t) da$ and $\dot{x}_d(t) = \int \lambda_d(a, t) \rho(a, t) da$. Also $x_b(t)$ is known from $x_b(0) = x(0) = \int \rho(a, 0) da$.

Now $\dot{x}_b(t) = \int \lambda_b(a, t) \rho(a, t) da$ where now $\lambda_b(a, t)$ has to be determined. We thus need to solve the inverse problem

$$\dot{x}_b(t) = \int_0^t \lambda_b(a, t) \dot{x}_b(t - a) e^{-\bar{\Lambda}^d(a, t)} da + \int_t^\infty \lambda_b(a, t) \rho_0(a - t) e^{-\bar{\Lambda}^d(a, t)} da,$$

where the unknown function is $\lambda_b(a, t)$. Suppose this inverse problem was solved. Then $(x_b(t), x_d(t), x(t))$ constitute a solution consistent with (24) and the rates $\lambda_b(a, t), \lambda_d(a, t)$.

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